Implications of Experiments on the Weak Undular Bore

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The behavior of the nonlinear dispersive waves observed behind weak undular bores is studied using available experimental information. Calculations on cnoidal wave theory using the measured wavelengths and amplitudes lead to the unexpected result that in bore-fixed coordinates the energy and momentum of the flow under the waves are larger than in the upstream flow. This can only be due to the action in the experiments of the unsteady boundary layer on the channel bottom, for this boundary layer has a negative momentum thickness. Therefore, the dissipation near the front that is known to be a necessary condition for the existence of the steady periodic waves is shown to occur on the bottom of the channel.

I. INTRODUCTION

FOR 100 years, since the earliest experimental studies of bores and hydraulic jumps,[1] it has been known that there is a qualitative difference between the profiles of strong and weak bores. While strong bores are steep and are characterized by breaking and turbulence just downstream of the front, the weak bore is a more gradual transition consisting of a front followed by an extensive train of periodic waves (Fig. 1). The reason for this spectacular difference is still unknown, though it has long been speculated that it is associated with the requirement of shallow-water theory that the flow lose energy as it passes through the front. It is thought that the waves behind the weak bore in some sense store energy that must otherwise be dissipated by turbulence at the front, making it possible for the required dissipation to be distributed over a much larger distance by an alternative process.

On the basis of an analysis of the observations of these periodic waves, Keulegan and Patterson[2] suggested that they were cnoidal waves, a type of nonlinear dispersive wave. Benjamin and Lighthill,[3] investigating the problem further with the aid of a new interpretation of cnoidal wave theory, showed that it is possible to patch a steady train of cnoidal waves downstream of a front to a uniform upstream flow *only if there is a change at the front of either mass flux, momentum flux, or energy.* If there were no change, then the only possible steady wave would be a cnoidal wave of infinite wavelength, the solitary wave. Benjamin and Lighthill therefore concluded that in order to explain the occurrence of periodic waves there must still be a little dissipation (e.g., a little turbulence) at the front of weak bores. In fact, an approximate calculation indicated that an energy loss at the front of about 20% of the classical energy loss would be sufficient to explain the experimental results.

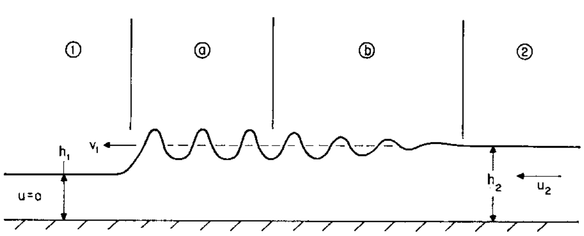

FIG. 1. The weak undular bore.

[1] H. Bazin, Mém. Prés. Acad. Sci., Paris **19**, 495 (1865).
[2] G. H. Keulegan and G. W. Patterson, J. Res. Natl. Bur. Std. **24**, 47 (1940).
[3] T. B. Benjamin and M. J. Lighthill, Proc. Roy. Soc. (London) **A224**, 448 (1954).

In this note we make a more careful calculation of the momentum and energy of the flow under the waves using cnoidal wave theory and data from the early experiments, and obtain results that show by what mechanism the dissipation near the front occurs. Our calculations explain why no turbulence is observed in the weakest bores and why none is needed. The conclusions do not depend strongly on the accuracy of the calculations or experiments; the qualitative behavior alone is sufficient to prove that the dissipation is in the boundary layer on the bottom of the channel.

II. EXPERIMENTS

There is an important distinction with regard to the channel boundary layer between experiments in which the bore propagates down a channel into water initially at rest (the initial-value problem) and experiments in which a stationary hydraulic jump between supercritical and subcritical steady flows is studied. In the former case the upstream "flow" can be made arbitrarily uniform. On the other hand, the upstream flow in the steady case is necessarily nonuniform and, for example, can be fully developed channel flow or turbulent. Consequently, boundary-layer effects are generally better defined in experiments with unsteady bores, and such experiments are preferable. The only published measurements of the waves behind such bores of sufficient accuracy and detail for the purposes of this note are those of Favre[4] obtained in 1935 from experiments in a $\frac{1}{2}$-m-wide by $\frac{1}{2}$-m-deep by 74-m-long open channel.

The idea that the flow can be described by patching a stationary train of cnoidal waves to the front is due to Favre's observation that the flow looks qualitatively as in Fig. 1, where the waves near the head of the bore [region (a)] seem to be of constant and steady amplitude.[5] As the bore propagates down the channel, waves continually form behind it, region (a) grows, and region (b) moves downstream relative to the front. Presumably, a long time after generating the bore, the flow downstream of the front consists of an infinite train of stationary waves. It is found from the measurements that the speed of the front, the average height h_2, and u_2, the flow velocity in region (2), accurately satisfy the classical jump conditions for bores. The amplitude of the waves near the head increases as the strength of the bore increases until, at a critical value $F_1 = 1.21$ [$F_1 = v_1(gh_1)^{-\frac{1}{2}}$ is the bore Froude number], breaking occurs at the front. Thereafter, the amplitude of the waves decreases and the amount of breaking increases.

III. CALCULATIONS

The accuracy of Favre's measurements is such that they can be used to examine the implications of cnoidal wave theory in detail: Three measured properties of the waves can be used to calculate the flux of mass, momentum and energy (in front-fixed coordinates),[6] assuming cnoidal waves, and the results can be compared, say, with the known values in the uniform flow upstream. For the three measured properties we use the depths under the crests and troughs and the wavelength. Unfortunately, all three of these quantities were measured in only six of Favre's runs.

Normalized slightly differently than in Ref. 3 in order to make the dependence on mass flow explicit, the cnoidal wave equation can be written

$$\tfrac{1}{3}q^2(dw/dz)^2 + w^2 - 3rw^2 + 3sw - q^2 = 0, \qquad (1)$$

where $w = \eta/h_1$ is the nondimensional depth, $z = x/h_1$ is the horizontal coordinate, $q = Q/c_1h_1$, $r = 2R/3c_1^2$, $s = 2S/3c_1^2h_1$, and $c_1 = (gh_1)^{\frac{1}{2}}$. Q, R, and S are the mass flow, the energy per unit mass, and the momentum flow rate corrected for pressure forces (Ref. 3), respectively. For a uniform flow (e.g., upstream),

$$\begin{aligned} &Q_1 = v_1h_1, \quad R_1 = \tfrac{1}{2}v_1^2 + gh_1, \quad S_1 = v_1^2h_1 + \tfrac{1}{2}gh_1^2, \\ &q_1 = F_1, \qquad r_1 = \tfrac{1}{3}F_1^2 + \tfrac{2}{3}, \qquad s_1 = \tfrac{2}{3}F_1^2 + \tfrac{1}{3}. \end{aligned} \qquad (2)$$

Now, if the roots of the cubic in Eq. (1) are w_1, w_2, and w_3 in descending order, the depths at crest and trough are w_1 and w_2, respectively (Ref. 3), and the wavelength λ is given by

$$\Lambda \equiv \frac{\lambda}{h_1} = 4\left[\frac{w_1w_2}{3}\left(\frac{w_1}{w_1 - w_2}k^2 - 1\right)\right]^{\frac{1}{2}}K(k), \qquad (3)$$

where K is the complete elliptic integral of the first kind with modulus $k = [(w_1 - w_2)/(w_1 - w_3)]^{\frac{1}{2}}$. Given w_1, w_2, and Λ, Eq. (3) gives k or w_3, and then q, r, and s can be calculated from

$$3r = w_1 + w_2 + w_3,$$

$$3s = w_1w_2 + w_2w_3 + w_1w_3, \qquad q^2 = w_1w_2w_3.$$

The corresponding quantities in the upstream flow, q_1, r_1, and s_1 are functions of the bore velocity F_1 only [Eqs. (2)]. Unfortunately, in the particular runs considered here, Favre did not measure the bore velocity. It turns out that, for the purposes of the present work, it is best to calculate the velocity from the conservation of mass, i.e., $F_1 = q_1 = q$, where q is calculated from w_1, w_2, and Λ, as outlined above. In our calculations we take for w_1 and w_2 the average of the values measured at the first three crests and troughs, respectively, behind the front.

IV. RESULTS

In cnoidal wave theory the modulus k varies between the limits 0 for sinusoidal waves and 1 for the solitary wave. Thus, the variation of k with bore strength gives an indication of the qualitative changes of wave form that occur. In fact, k^2 is essentially $a\lambda^2/h_2^3$, a parameter that in shallow-water

[4] H. Favre, *Étude théorique et expérimentale des ondes de translation dans les canaux découverts* (Dunod et Cie., Paris, 1935).

[5] It has not been established that in all cases weak bores look as in Fig. 1. For example, it is not known if a distinct region (a) can be observed behind bores in smaller channels, or whether a minimum channel length is required for the development of such a region.

[6] In the following we consider the problem in bore-fixed coordinates. The upstream velocity is v_1, and the velocity in region (2), $v_2 = v_1 - u_2$, is given by the classical bore jump conditions. In this coordinate system the flow in regions (1), (a) and (2) is steady.

TABLE I. Results from the Favre[a] experiments using cnoidal wave theory.

Run no.	h_1(cm)	$\frac{h_2}{h_1}$	w_1	w_2	Λ	k^2	w_3	$q = F_1$	r/r_1	s/s_1	$\frac{r - r_2}{r_1 - r_2}$
21	10.78	1.080	1.155	1.020	11.039	0.6656	0.9528	1.060	1.0017	1.0016	27.0
22	10.73	1.140	1.259	1.049	9.413	0.6718	0.9470	1.119	1.0013	1.0014	3.3
23	10.79	1.230	1.443	1.030	8.434	0.7955	0.9235	1.171	1.0071	1.0067	5.9
24	10.74	1.281	1.561	1.031	8.007	0.8185	0.9132	1.212	1.0103	1.0094	5.0
26	10.75	1.375	1.603	1.249	5.861	0.4388	0.7972	1.263	1.0147	1.0199	4.2
29	10.61	1.501	1.629	1.465	5.090	0.1762	0.6979	1.290	1.0345	1.0496	6.9

[a] See Ref. 2.

theory measures the relative magnitudes of nonlinear steepening and dispersion (a = wave height from trough to crest). Actually, $a\lambda^2/h_1^3 = \frac{16}{3}q^2k^2$. Figure 2 is a plot of k^2 and a/h_2 vs F_1 calculated from Favre's data (cf. Table I). It can be seen that as the strength of the bore increases the waves carry more energy by getting larger and longer, tending to become a train of nearly solitary waves. However, the amplitude of the first wave above the undisturbed flow ultimately becomes so large ($a/h_1 \doteq 0.56$ at $F_1 = 1.21$) that it breaks. Thereafter, the amplitude of the waves decreases, and they tend to become sinusoidal again.

Comparison with the upstream momentum and energy is made by plotting S/S_1 and R/R_1 vs F_1 in Fig. 3. These quantities are not <1 as originally expected; the calculations show that there is actually an *increase* at the front of both momentum and energy. At first glance this result is rather surprising, but, in fact, it is consistent with the idea that changes in momentum and energy are due to the action of viscosity near the bottom of the channel. For, the boundary layer generated under the bore in these experiments is much like that in the flow behind a shock wave moving down a tube into gas at rest, the familiar shock-tube boundary layer, which is well known to have negative displacement and momentum thicknesses. In shock-fixed coordinates the wall moves faster than the free-stream flow and therefore puts momentum (and kinetic energy) into the fluid. The magnitude of the increases indicated in Fig. 3 is not inconsistent with the fact that the boundary-layer thickness one wavelength behind the front in Favre's experiments is probably at least several per cent of h_2 (the Reynolds number c_1h_1/ν is about 35 000).[7] At any rate, because of the unique feature of this unsteady boundary layer, the sign of the observed energy change indicates unequivocally that boundary-layer action (rather than some other dissipative process) determines the properties of the waves. Unfortunately, Favre's experiments were performed at only one depth h_1, so the effect of changes of the scale of the boundary layer can not be determined.

It is notable that, even at high Froude numbers, when the wave amplitude is small, the wave energy is still quite large, and R shows no tendency to approach R_2. In these experiments the increase of energy was always several times larger than the decrease predicted by the classical-bore jump con-

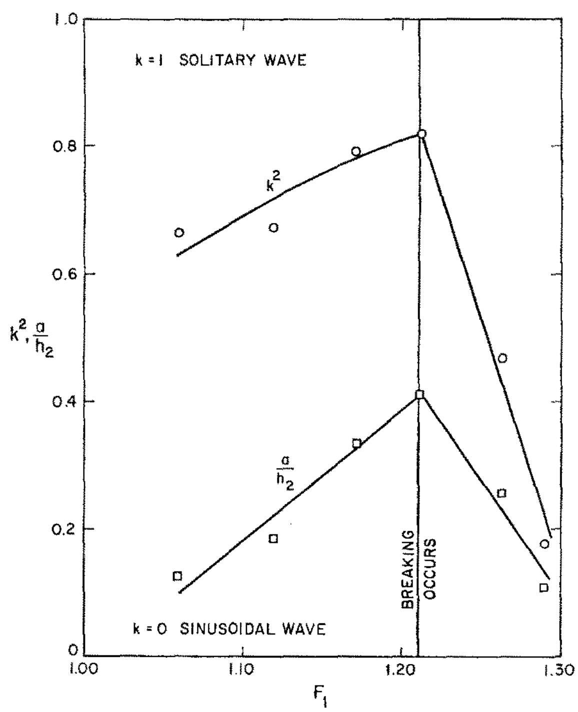

FIG. 2. Variation of the modulus k and the wave amplitude with bore strength.

[7] Furthermore, if it is assumed that the measured increase of momentum can be roughly accounted for by a Chezy resistance law acting for, say, one wavelength in the streamwise direction, then the average value of the von Mises resistance coefficient implied by runs 21–26 is $f \approx 0.19$, in surprising agreement with the value $f \doteq 0.18$ for concrete (the material used in Favre's channel), calculated from Eq. (194) of H. Rouse, *Fluid Mechanics for Hydraulic Engineers* (McGraw-Hill Book Company, Inc., New York, 1938), p. 279.

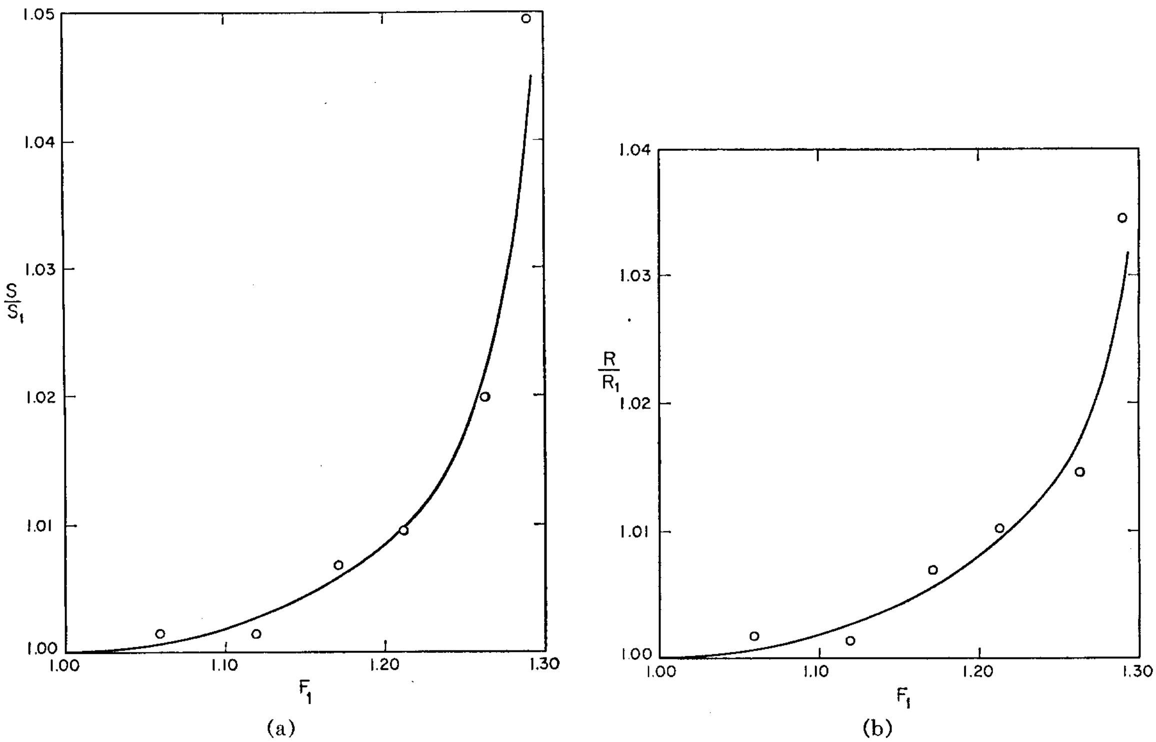

FIG. 3. Comparison of the flow under the waves with the upstream flow: (a) Momentum flux; (b) Flow energy.

ditions (cf. the last column of Table I). Because of this fact, it is impossible that conditions in region (2) can exactly satisfy the jump conditions. A complete theory of this problem, if available, would predict these deviations, just as shock-tube boundary-layer theory predicts departures from ideal shock-tube flow. Indeed, the boundary layer generated by a weak bore corresponds, in a sense, to the boundary layer behind a weak-to-moderate-strength shock wave in a shock tube. Studies of shock-tube test time have shown that in this case there can be a very strong interaction between the boundary layer and the outer flow, the layer causing large streamwise pressure gradients and, in turn, being influenced by these gradients. Similarly, we have shown that the boundary layer generated by a bore exerts some causative influence on the waves; and it must in turn be strongly affected by the large periodic pressure gradients under the waves.

V. CONCLUSIONS

Calculations on cnoidal wave theory using data from experiments on weak bores propagating down a channel into water at rest have led to the unexpected result that in front-fixed coordinates the flow energy immediately behind the bore is actually larger than ahead of it. This is evidently due to the effect of the unsteady boundary layer on the bottom of the channel, and the waves behind the bore constitute a very sensitive measure of the nature of this boundary layer.

Interest in this problem has been stimulated recently because of its similarity to the problem of the collisionless shock wave in plasma physics. The question of the dissipation necessary to explain the occurrence of steady cnoidal waves behind the bore is related to similar questions concerning the existence of steady shock solutions of the plasma equations. Our conclusion that dissipation in the channel boundary layer is decisive in the bore problem sheds no light on the elusive dissipative mechanism in the collisionless shock wave.

Further experiments exploring some of the questions raised in this study are being carried out in a new water channel at the Graduate Aeronautical Laboratories, California Institute of Technology.

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